Distortion of normalized quasiconformal mappings

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Abstract: Quasiconformal homeomorphisms of the whole space of $\mathbb{R}^n$, onto itself normalized at one or two points are studied. In particular, the case when the maximal dilatation tends to 1 is in the focus. Our methods make use of an asymptotically sharp bound for quasisymmetry and generalized Bernoulli type inequality. In addition, we prove sharp result for the behavior of the quasihyperbolic metric under quasiconformal maps.

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1 Introduction

For $n \geq 2, K \geq 1$, let

$$QC_K(\mathbb{R}^n) = \{f \,:\, \mathbb{R}^n \to \mathbb{R}^n \,:\, f \text{ is } K- \text{ quasiconformal } \}\,.$$

Here and through the paper we use the standard definition of K-quasiconformality from [V1]. It is a well-known basic fact that a map $f \in QC_K(\mathbb{R}^n)$ has a homeomorphic extension to ∞ with $f(\infty) = \infty$ (in fact, this can be also seen from Proposition 2.5 below). In this way, f is defined in the Möbius space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. Without further remark we always assume that our maps are extended in this way. For the sake of convenience, we will consider a subclass of $QC_K(\mathbb{R}^n)$ consisting of maps normalized at two finite points as follows

$$M_K(\mathbb{R}^n) = \{f \in QC_K(\mathbb{R}^n) \colon f(x) = x, \quad \forall x \in \{0, e_1, \infty\}\}\,.$$

In his classical work [T] O. Teichmüller studied the class $M_K(\mathbb{R}^2)$ and proved the following inequality for the hyperbolic metric s_G of $G = \mathbb{R}^2 \setminus \{0, e_1\}$: If $f \in M_K(\mathbb{R}^2)$, then the sharp inequality

$$K \geq \exp(s_G(x, f(x)))$$

holds for all $x \in G$. For the definition of the hyperbolic metric see [KL].

This result may be considered as a stability result: $f(x)$ is contained in the ball $B_{s_G}(x, \log K)$ of the metric s_G centered at the point x and with the radius $\log K$. In particular, for $K \to 1$, the radius tends to 0 .

On one hand this result is sharp, on the other hand the information it provides is implicit. Indeed the geometric structure of the balls $B_{s_G}(x, r)$ has

not been carefully studied to our knowledge. Furthermore, for a general plane domain D, it is a basic fact that the balls $B_{s_D}(z,t)$ depend very much on the boundary ∂D, the center z and the radius t, but quantitative estimates are hard to find for nonspecialists in the literature. For instance, it may happen that $B_{s_D}(z_1,r)$ is topologically equivalent to the unit disk whereas $B_{s_D}(z_2,r), z_2 \neq z_1$, is equivalent to an annulus.

Therefore it seems to be a natural question to study Teichmüller's result in the context of metric spaces with a more concrete metric than the hyperbolic metric. An example of such a metric is the distance ratio metric or the j-metric studied below. Our goal here is to extend, at least partially, Teichmüller's result to $\mathbb{R}^3$ because a rotation around the e_1-axis leaves the whole e_1-axis and in particular the triple $\{0,e_1,\infty\}$ fixed, we see that for a fixed $x \in \mathbb{R}^3 \setminus \{0,e_1\}$ and $K \geq 1$

$$V_K(x) = \{f(x) : f \in M_K(\mathbb{R}^3)\}$$

has rotational symmetry, it is a solid of revolution with the e_1-axis as the symmetry axis.

In Section 2 we prove some preliminary results. Bernoulli type inequalites that will be used in the proof the main results will be given in Section 3.

Our first main result, Theorem 5.4 provides information about the size of the set $V_K(x)$ for $K>1$ close to 1. The proof of this theorem combines a number of ideas. The first idea is to show that the images of both of the spheres $S^{n-1}(0,|x|)$ and $S^{n-1}(e_1,|x-e_1|)$ are contained in spherical ring domains, centered at 0 and e_1, respectively, with a good control of the inner and outer radii in each case. Therefore $V_K(x)$ is a subset of their intersection and it remains to estimate the size of the intersection in terms of radii. For this purpose we prove some new elementary inequalities.

Our second main result, Theorem 5.5 deals with the behavior of so called distance ratio metric. Also here an asymptotically sharp result is proved.

The *quasihyperbolic metric* of G is defined by the quasihyperbolic length minimizing property

$$k_G(x,y) = \inf_{\gamma \in \Gamma(x,y)} \ell_k(\gamma), \quad \ell_k(\gamma) = \int_\gamma \frac{|dz|}{d(z)},$$

where $\ell_k(\gamma)$ is the quasihyperbolic length of γ (cf. [GP]) and $d(z)$ stands for the distance $d(z,\partial G)$ of $z \in G$ to the boundary. Gehring and Osgood proved in [GO] that quasihyperbolic metric is quasiinvariant under K-quasiconformal maps $f : G \to f(G)$, $G, f(G) \subset \mathbb{R}^n$. It can be easily shown that the quasihyperbolic metric is not invariant under Möbius transformations of the unit ball onto itself and hence their result cannot be sharp

when $K \to 1$. Our third main result, Theorem 5.7, provides a refined version of [GO, Theorem 3], which is sharp when $K \to 1$ for the case of the particular domain $\mathbb{R}^n \setminus \{0\}$. This result is new also in the case $n = 2$.

2 Preliminary results

When $s > 1$ for the *Grötzsch capacity* we use the notation $\gamma_n(s)$ as in [Vu1, p. 88]. Then for the planar case we have by [Vu1, p.66] $\gamma_2(s) = 2\pi/\mu(1/s)$, where

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)} \quad \text{and} \quad \mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}$$

for $r \in (0,1)$. We define for $r \in (0,1)$ and $K > 0$

$$\varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))}.$$

Let $\eta\colon [0,\infty) \to [0,\infty)$ be an increasing homeomorphism and $D, D' \subset \mathbb{R}^n$. A homeomorphism $f\colon D \to D'$ is η*-quasisymmetric* if

$$\frac{|f(a)-f(c)|}{|f(b)-f(c)|} \le \eta\left(\frac{|a-c|}{|b-c|}\right) \tag{2.1}$$

for all $a, b, c \in D$ and $c \neq b$. By [V2] K-quasiconformal mapping of the whole $\mathbb{R}^n$ is $\eta_{K,n}$-quasisymmetric with a control function $\eta_{K,n}$. Let us define the optimal control function by

$$\eta^*_{K,n}(t) = \sup\{|f(x)|\colon |x| \le t, f \in QC_K(\mathbb{R}^n), f(y) = y \text{ for } y \in \{0, e_1, \infty\}\}.$$

Vuorinen [Vu2, Theorem 1.8] proved an upper bound for $\eta^*_{K,n}(t)$, which was later refined in [AVV, Theorem 14.8] for $K \ge 1$ into the following form

$$\eta^*_{K,n}(t) \le \begin{cases} \eta^*_{K,n}(1)\varphi_{K,n}(t), & 0 < t < 1, \\ \exp(4K(K+1)\sqrt{K-1}), & t = 1, \\ \eta^*_{K,n}(1)\dfrac{1}{\varphi_{1/K,n}(1/t)}, & t > 1. \end{cases} \tag{2.2}$$

A simplified, but still asymptotically sharp upper bound for $\eta^*_{K,n}(t)$ can be written as follows

$$\eta^*_{K,n}(t) \le \begin{cases} \eta^*_{K,n}(1)\lambda_n^{1-\alpha}t^\alpha, & 0 < t \le 1, \\ \eta^*_{K,n}(1)\lambda_n^{\beta-1}t^\beta, & t > 1, \end{cases} \tag{2.3}$$

where $\alpha = K^{1/(1-n)}$ and $\beta = 1/\alpha$. Furthermore, by [Vu1, Lemma 7.50] we have the following estimation

$$\lambda_n^{1-\alpha} \le 2^{1-1/K} K \quad \text{and} \quad \lambda_n^{1-\beta} \ge 2^{1-K} K^{-K}. \tag{2.4}$$

2.5 Proposition. *Let $K \in (1,2]$, $f \in QC_K(\mathbb{R}^n)$, $f(x) = x$ for $x \in \{0, e_1\}$, $\alpha = K^{1/(1-n)}$ and $\beta = 1/\alpha$. Then*

$$\begin{aligned} &\frac{1}{c_3}|x|^\beta \le |f(x)| \le c_3|x|^\alpha, \quad \text{if } 0 < |x| \le 1,\\ &\frac{1}{c_3}|x|^\alpha \le |f(x)| \le c_3|x|^\beta, \quad \text{if } |x| > 1 \end{aligned}$$

for $c_3 = \exp(60\sqrt{K-1})$.

Proof. Since f is quasiconformal it is also $\eta^*_{K,n}$-quasisymmetric and by choosing $a = x$, $b = 0$ and $c = e_1$ in (2.1) we have $|f(x)| \le \eta^*_{K,n}(|x|)$. Similarly, selection $(a,b,c) = (e_1, 0, x)$ in (2.1) gives $|f(x)| \ge 1/\eta^*_{K,n}(1/|x|)$. Therefore

$$\frac{1}{\eta^*_{K,n}(1/|x|)} \le |f(x)| \le \eta^*_{K,n}(|x|) \tag{2.6}$$

for all $x \in \overline{\mathbb{R}}^n \setminus \{0\}$. Therefore by (2.3)

$$\begin{aligned} &\frac{1}{c_2}|x|^\beta \le |f(x)| \le c_1|x|^\alpha, && \text{if } 0 < |x| < 1,\\ &\frac{1}{\eta^*_{K,n}(1)} \le |f(x)| \le \eta^*_{K,n}(1), && \text{if } |x| = 1,\\ &\frac{1}{c_1}|x|^\alpha \le |f(x)| \le c_2|x|^\beta, && \text{if } |x| > 1, \end{aligned}$$

for $c_1 = \eta^*_{K,n}(1)\lambda_n^{1-\alpha}$ and $c_2 = \eta^*_{K,n}(1)\lambda_n^{\beta-1}$. We can estimate $\max\{c_1, c_2\} \le c_3 = \exp(60\sqrt{K-1})$ for $K \in (1,2]$. $\square$

2.7 Lemma. *If $0 < a < 1 < b$, $q = \sqrt{\frac{b-1}{1-a}}$ and $m \ge \max\{q, q^{-1}\}$, then*

$$mt^a - t \ge t - \frac{t^b}{m} \tag{2.8}$$

holds for $0 < t \le 1$ and

$$mt^b - t \ge t - \frac{t^a}{m} \tag{2.9}$$

holds for $t \ge 1$.

Proof. (1) With $f(t) = mt^{a-1} + \frac{t^{b-1}}{m}$ we see that the first inequality is equivalent to $f(t) \geq 2$ for $0 < t \leq 1$. By differentiating we conclude that the function f is decreasing for

$$0 < t \leq \left(\frac{m^2(1-a)}{b-1} \right)^{1/(b-a)} \equiv t_0.$$

Therefore the function f has a minimum at t_0. By the choice of m we see that $t_0 \geq 1$ and hence for all $t \in (0,1]$

$$f(t) \geq f(1) = m + m^{-1} \geq 2.$$

The first inequality (2.8) is proved and the proof for (2.9) is similar. □

2.10 Lemma. *Let $n \geq 2$, $K > 1$, $\alpha = K^{1/(1-n)}$, $\beta = 1/\alpha$ and $c_3 = \sqrt{\beta}$. For $t \in (0,1)$*

$$c_3 t^\alpha - t \geq t - \frac{t^\beta}{c_3} \tag{2.11}$$

and for $t > 1$

$$c_3 t^\beta - t \geq t - \frac{t^\alpha}{c_3}. \tag{2.12}$$

Moreover, both (2.11) and (2.12) hold for all constants $c_4 \geq c_3$, e.g. for $c_4 = e^{60\sqrt{K-1}}$.

Proof. For the application of Lemma 2.7 we observe that

$$\frac{\beta - 1}{1 - \alpha} = \beta > 1.$$

Now the proof follows from Lemma 2.7 and the inequalities

$$e^{60\sqrt{K-1}} \geq e^{\sqrt{\beta-1}} \geq \sqrt{\beta}\,.$$

Note that that the function $h(t) = m \max\{t^\alpha, t^\beta\} + \min\{t^\alpha, t^\beta\}/m$ is increasing in m when t is fixed. □

3 Bernoulli type inequalities

In this section we introduce some Bernoulli type inequalities. To prove our main result we need the following lemma, which is new to our knowledge. To some extent it is similar to the Bernoulli type inequalities in [AVV, 1.58(30)]. For $a = 1 = b$ part (7) of the next lemma coincides with the usual Bernoulli inequality [M, p. 34(4)].

3.1 Lemma. *Let* $0 < a \le 1 \le b$*. Then, with* $u = \log^{1-a} 2$, $v = \log^{1-b} 2$,

(1) the function

$$f_1(t) = \frac{\log(1+t)}{\log(1+t^a)}$$

is increasing on $(0,\infty)$ *with range* $(0, 1/a)$*.*

(2) for $t \in (0,\infty)$

$$u \le f_2(t) < 1,$$

where

$$f_2(t) = \frac{\log(1+t^a)}{\log^a(1+t)}.$$

The function $f_2(t)$ *is decreasing on* $(0,1)$ *and increasing on* $(1,\infty)$ *with* $f_2(1) = u$*.*

(3) for $t \in (0,\infty)$

$$f_3(t) = \frac{\log(1+t^b)}{\log^b(1+t)} \le v.$$

The function $f_3(t)$ *is increasing on* $(0,1)$ *and decreasing on* $(1,\infty)$ *with* $f_3(1) = v$*.*

(4) the function

$$f_4(t) = \frac{\log(1+t^b)}{\log(1+t)}$$

is increasing on $(0,\infty)$ *with range* $(0, b)$*.*

(5) for $\varphi(t) = \max\{t^a, t^b\}$ *and* $t \in (0, e-1]$

$$u \log(1+\varphi(t)) \le \varphi(\log(1+t)) \le u^{-1} \log(1+\varphi(t)).$$

(6) for $\varphi(t) = \max\{t^a, t^b\}$ *and* $t > 0$

$$\log(1+\varphi(t))/c_5 \le \varphi(\log(1+t)) \le c_5 \log^b(1+\varphi(t)),$$

where $c_5 \to 1$ *as* $\max\{b, 1/a\} \to 1$*,*

(7) for $\varphi(t) = \max\{t^a, t^b\}$ *and* $c > 1$

$$\log(1+c\varphi(t)) \le \begin{cases} c\log^a(1+t), & 0 < t < 1, \\ cb\log(1+t), & t \ge 1. \end{cases}$$

(8) for $\varphi(t) = \max\{t^a, t^b\}$ *and* $s, t > 0$

$$2^{1-b} \leq \frac{\varphi(s) + \varphi(t)}{\varphi(s+t)} \leq 2^{1-a}.$$

Proof. *(1)* We will first show that $f_1(t)$ is an increasing function. By a straightforward computation

$$f_1'(t) = \frac{1}{\log^2(1+t^a)} \left(\frac{\log(1+t^a)}{1+t} - \frac{a\log(1+t)}{t+t^{1-a}} \right)$$

and $f_1'(t) \geq 0$ is equivalent to

$$ag(t) \leq g(t^a) \tag{3.2}$$

for $g(t) = (1+1/t)\log(1+t)$. By differentiation we see that $g(t)$ is increasing and therefore (3.2) holds for $t \in (0,1]$.

Let us assume $t > 1$. We will show that the function $h(t) = g(t^a) - ag(t)$ is decreasing. We obtain

$$h'(t) = \frac{a}{t^{2+a}} \left(t^a \log(1+t) - t\log(1+t^a) \right)$$

and $h'(t) \leq 0$ is equivalent to $s(t) \leq s(t^a)$ for $s(t) = \frac{\log(1+t)}{t}$. By [M, p. 273, 3.6.18] $t/(1+t) \leq \log(1+t)$ and therefore

$$s'(t) = \frac{\frac{t}{1+t} - \log(1+t)}{t^2} \leq 0$$

and implying $s(t) \leq s(t^a)$. We conclude that $f_1(t)$ is an increasing function on $(0,\infty)$.

By the l'Hospital Rule

$$\lim_{t\to 0} f_1(t) = \lim_{t\to 0} \frac{t^{1-a}(1+t^a)}{a(1+t)} = 0$$

and

$$\lim_{t\to\infty} f_1(t) = \lim_{t\to\infty} \frac{t^{1-a}(1+t^a)}{a(1+t)} = \frac{1}{a}$$

and the assertion follows.

(2) We will show that $f_2(t)$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$. We have

$$f_2'(t) = \frac{a}{\log^a(1+t)} \left(\frac{1}{t+t^{1-a}} - \frac{\log(1+t^a)}{(1+t)\log(1+t)} \right)$$

and $f_2'(t) \le 0$ is equivalent to

$$g(t) \le g(t^a) \tag{3.3}$$

for $g(t) = (1 + 1/t)\log(1 + t)$. The function $g(t)$ is increasing on $(0, \infty)$ because $g'(t) = (t - \log(1 + t))/t^2 \ge 0$. Therefore inequality (3.3) is true and $f_2'(t) \le 0$ for $t \in (0, 1]$. Similarly, $f_2'(t) \ge 0$ for $[1, \infty)$. Now $f_2(t)$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$. Therefore

$$f_2(t) \le \max\left\{\lim_{t\to 0} f_2(t), \lim_{t\to\infty} f_2(t)\right\} = 1$$

and

$$f_2(t) \ge f_2(1) = u.$$

(3) Follows from *(2)* by choosing $b = 1/a$.

(4) We show that $f_4(t)$ is an increasing function on $(0, \infty)$. Since

$$f_4'(t) = \frac{1}{\log^2(1+t)}\left(\frac{b\log(1+t)}{t+t^{1-b}} - \frac{\log(1+t^b)}{1+t}\right)$$

the inequality $f_4'(t) \ge 0$ is equivalent to $bg(t) \ge g(t^b)$ for $g(t) = (1 + 1/t)\log(1 + t)$. It holds true by proof of *(1)*.

By the l'Hospital Rule

$$\lim_{t\to 0} f_4(t) = \lim_{t\to 0} \frac{b(1+t)t^{b-1}}{(1+t^b)} = 0$$

and

$$\lim_{t\to\infty} f_4(t) = \lim_{t\to\infty} \frac{b(1+t)t^{b-1}}{(1+t^b)} = b$$

and the assertion follows.

(5) If $t \in (0, 1]$ then by *(2)*

$$\log(1 + \varphi(t)) = \log(1 + t^a) \le \log^a(1 + t) = \varphi(\log(1 + t))$$

and

$$\varphi(\log(1 + t)) = \log^a(1 + t) \le u^{-1}\log(1 + t^a) = u^{-1}\log(1 + \varphi(t)).$$

If $t \in [1, e - 1]$ then by *(3)*

$$\log(1 + \varphi(t)) = \log(1 + t^b) \le v\log^b(1 + t) \le v\log^a(1 + t) = v\varphi(\log(1 + t))$$

and by *(2)*

$$\varphi(\log(1+t)) = \log^a(1+t) \le u^{-1}\log(1+t^a) \le u^{-1}\log(1+t^b) = u^{-1}\log(1+\varphi(t)).$$

Since $\max\{1/u, 1/v\} = 1/u$ the assertion follows.

(6) By *(5)* we may assume $t \geq e - 1$. Now by *(3)*

$$\log(1 + \varphi(t)) = \log(1 + t^b) \leq v \log^b(1 + t) = v\varphi(\log(1 + t))$$

and

$$\varphi(\log(1 + t)) = \log^b(1 + t) \leq \log^b(1 + t^b) = \log^b(1 + \varphi(t)).$$

We may choose $c_5 = \max\{1/u, v\}$ and the assertion follows.

(7) The assertion follows since by the Bernoulli inequality [M, p. 34(4)] and *(2)*

$$\log(1 + ct^a) \leq c \log(1 + t^a) \leq c \log^a(1 + t)$$

and by the Bernoulli inequality [M, p. 34(4)] and *(4)*

$$\log(1 + ct^b) \leq c \log(1 + t^b) \leq cb \log(1 + t).$$

(8) Follows from [AVV, 1.58 (27) p. 340]. $\square$

3.4 Lemma. *(1) For $0 < a < b < 1$ the function*

$$f_5(t) = \frac{b^t - a^t}{t}$$

is decreasing on $(0, \infty)$.

(2) For $a > 0$ the function

$$f_6(t) = t \log \frac{1 + a/t}{1 - a/t}$$

is decreasing on (a, ∞).

Proof. *(1)* We show that

$$f_5'(t) = \frac{1}{t^2}(a^t - b^t + tb^t \log b - ta^t \log a) \leq 0$$

which is equivalent to

$$g(b) \leq g(a) \tag{3.5}$$

for $g(x) = x^t(t \log x - 1)$. Since $g'(x) = t^2 x^{a-1} \log x \leq 0$ and $a < b$ the inequality (3.5) holds true and the assertion follows.

(2) We show that

$$f_6'(t) = \log \frac{t + a}{t - a} - \frac{2at}{t^2 - a^2} \leq 0.$$

Since

$$f_6''(t) = \frac{4a^3}{(t - a)^2(t + a)^2} \geq 0$$

$f_6'(t)$ is increasing and we have $f_6'(t) \leq \lim_{t\to\infty} f'(t) = 0$. $\square$

4 Diameter estimate

In this section we will consider K- quasiconformal mapping $f\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ with $f(y) = y$ for $y \in \{0, e_1, \infty\}$ and our goal is to find an upper bound for $|f(x)-x|$ or similar quantities in terms of K and n, when $|x| \le 2$ and $K > 1$ is small enough.

4.1 Description of the goal. Fix $x \in \mathbb{R}^n \setminus \{0, e_1\}$ and assume that $|x| - \varepsilon \le |f(x)| \le |x| + \varepsilon$ and $|x - e_1| - \varepsilon \le |f(x) - e_1| \le |x - e_1| + \varepsilon$ for $\varepsilon \in (0, (1 - ||x| - |x - e_1||)/2)$. Now

$$|f(x) - x| \le \frac{\operatorname{diam}(A)}{2}, \tag{4.2}$$

where

$$A = A(0, |x|+\varepsilon, |x|-\varepsilon) \cap A(e_1, |x-e_1|+\varepsilon, |x-e_1|-\varepsilon) \cap \{z \in \mathbb{R}^n\colon z_i = 0,\ i \ge 3\}$$

and

$$A(z, R, r) = B^n(z, R) \setminus \overline{B}^n(z, r).$$

We will now find upper bounds for $\operatorname{diam}(A)$.

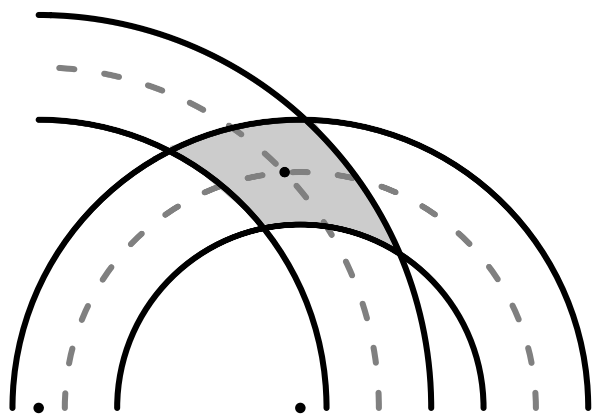

Figure 1: The shaded region is the cross-section of the set A in (4.2) with the plane that contains both 0 and e_1.

In the next theorem we give a quantitative sufficient condition for the inequality (4.2).

4.3 Lemma. *Fix $x \in \mathbb{R}^n \setminus \{0, e_1\}$ and $\varepsilon \in (0, (1 - ||x| - |x - e_1||)/2)$. Then*

$$|x| - \varepsilon \le |f(x)| \le |x| + \varepsilon \quad \textit{and} \quad |x - e_1| - \varepsilon \le |f(x) - e_1| \le |x - e_1| + \varepsilon$$

for

$$1 < K \le \min\left\{\left(\frac{\log(\varepsilon/2+1)}{62}\right)^2 + 1, 2\right\}.$$

Proof. We show first that $|x| - \varepsilon \le |f(x)| \le |x| + \varepsilon$ holds. Let us denote $l(x) = c_3^{-1}\max\{|x|^\alpha, |x|^\beta\}$ and $u(x) = c_3 \max\{|x|^\alpha, |x|^\beta\}$ where $c_3 = e^{60\sqrt{K-1}}$.

We will first consider the case $0 < |x| < 1$. By Lemma 2.10

$$\begin{aligned}
\max\{u(x) - |x|, |x| - l(x)\} &= \max\left\{c_3|x|^\alpha - |x|, |x| - \frac{1}{c_3}|x|^\beta\right\}\\
&= c_3|x|^\alpha - |x|\\
&\le \exp(60\sqrt{K-1})|x|^\alpha - |x|\\
&\le \exp(60\sqrt{K-1})|x|^{1/K} - |x|\\
&\le \exp(60\sqrt{K-1}) - 1.
\end{aligned}$$

Now $\exp(60\sqrt{K-1}) - 1 \le \varepsilon$ is equivalent to

$$K \le \left(\frac{\log(\varepsilon+1)}{60}\right)^2 + 1. \tag{4.4}$$

If $|x| = 1$, then

$$\max\{u(x) - |x|, |x| - l(x)\} = c_3 - 1$$

and therefore we want $\exp(60\sqrt{K+1}) - 1 \le \varepsilon$ for $K \in (1, 2]$, which is equivalent to

$$K \le \left(\frac{\log(\varepsilon+1)}{60}\right)^2 + 1. \tag{4.5}$$

Let us then consider the case $1 < |x| < 2$. By Lemma 2.10

$$\begin{aligned}
\max\{u(x) - |x|, |x| - l(x)\} &= \max\left\{c_3|x|^\beta - |x|, |x| - \frac{1}{c_3}|x|^\alpha\right\}\\
&= c_3|x|^\beta - |x|\\
&\le \exp(60\sqrt{K-1})|x|^\beta - |x|\\
&\le \exp(60\sqrt{K-1})|x|^K - |x|\\
&\le |x|(\exp(60\sqrt{K-1})|x|^{K-1} - 1)\\
&\le 2(\exp(60\sqrt{K-1} + (K-1)\log|x|) - 1)\\
&\le 2(\exp(62\sqrt{K-1}) - 1).
\end{aligned}$$

Now $2(\exp(62\sqrt{K-1})-1) \le \varepsilon$ is equivalent to

$$K \le \left(\frac{\log(\varepsilon/2+1)}{62}\right)^2 + 1. \tag{4.6}$$

By combining (4.4), (4.5) and (4.6) we have

$$|x| - \varepsilon \le |f(x)| \le |x| + \varepsilon$$

for

$$\begin{aligned} K &\le \min\left\{\left(\frac{\log(\varepsilon+1)}{60}\right)^2 + 1, 2, \left(\frac{\log(\varepsilon/2+1)}{62}\right)^2 + 1\right\} \\ &= \min\left\{\left(\frac{\log(\varepsilon/2+1)}{62}\right)^2 + 1, 2\right\} \end{aligned}$$

and thus $|x|-\varepsilon \le |f(x)| \le |x|+\varepsilon$. Similar argument also implies $|x-e_1|-\varepsilon \le |f(x)-e_1| \le |x-e_1|+\varepsilon$ and the assertion follows. $\square$

4.7 Theorem. *Fix $x \in \mathbb{R}^n \setminus \{0, e_1\}$, $\varepsilon \in (0, (1 - ||x| - |x-e_1||)/2)$ and let A be as in (4.2). Then*

$$\operatorname{diam}(A) \le \sqrt{\varepsilon}4(\min\{|x|, |x-e_1|\} + 1).$$

Proof. Let us assume $|x| \le |x-e_1|$. Now $\operatorname{diam}(A)$ is maximal when $|x-e_1|/|x|$ is maximal. Therefore we may assume $x = -s$, where $s > 0$. Denote $y = S^1(0, |x|+\varepsilon) \cap S^1(e_1, |x-e_1|-\varepsilon)$. Area of the triangle $\triangle e_1 0 y$ is $(\operatorname{Im} y)/2$ and by Heron's formula

$$\frac{\operatorname{Im} y}{2} = \sqrt{p(p-1)(p-|x|-\varepsilon)(p+\varepsilon-|x|-1)}, \tag{4.8}$$

where $p = |x|+1$. By (4.8) and the assumption $\varepsilon < 1$ we have

$$\operatorname{Im} y = 2\sqrt{(|x|+1)|x|(1-\varepsilon)\varepsilon} \le 2\sqrt{\varepsilon}(|x|+1)$$

and the assertion follows since $\operatorname{diam}(A) \le 2\operatorname{Im} y$. $\square$

5 The main results

In this section we will use the diameter estimates to obtain results for the following two metrics.

The spherical (chordal) metric in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ is defined by

$$q(x,y) = \begin{cases} \dfrac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, & x \neq \infty \neq y, \\ \dfrac{1}{\sqrt{1+|x|^2}}, & y = \infty. \end{cases}$$

and the *distance ratio metric* or *j-metric* in a proper subdomain G of the Euclidean space $\mathbb{R}^n$, $n \geq 2$, is defined by

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right),$$

where $d(x)$ is the Euclidean distance between x and ∂G.

For convenience of the reader we recall some basic properties of the quasihyperbolic metric. For a given pair of points $x, y \in G$, the infimum in the definition of k_G is always attained [GO], i.e., there always exists a quasihyperbolic geodesic $J_G[x,y]$ which minimizes the quasihyperbolic length, $k_G(x,y) = \ell_k(J_G[x,y])$ and furthermore the distance is additive on the geodesic: $k_G(x,y) = k_G(x,z) + k_G(z,y)$ for all $z \in J_G[x,y]$.

5.1 Lemma. *([KSV, Lemma 2.3 (3)], [GP, Lemma2.1]). Fix $\lambda \in (0,1)$. For $x, y \in G = \mathbb{R}^n \setminus \{0\}$ with $|x-y| \leq \lambda |x|$ we have*

$$j_G(x,y) \leq k_G(x,y) \leq c_1(\lambda) j_G(x,y)$$

with $c_1(\lambda) = 1 + \lambda$.

With the weaker constant $1/(1-\lambda)$ in place of $c_1(\lambda)$ the upper bound in Lemma 5.1 occurs in [Vu1, Lemma 3.7 (2)].

Only in rare special cases there is a formula for the quasihyperbolic distance of two points. One such case is when $x, y \in G = \mathbb{R}^n \setminus \{0\}$. Martin and Osgood [MO] proved that

$$k_G(x,y) = \sqrt{\log^2 \frac{|x|}{|y|} + \left(2 \arcsin\left(\frac{1}{2}\left(\left|\frac{x}{|x|} - \frac{y}{|y|}\right|\right)\right)\right)^2}, \tag{5.2}$$

for all $x, y \in G$.

5.3 Theorem. *Let $x \in \mathbb{R}^n \setminus \{0, e_1\}$, A be as in* (4.2) *and $K \in (1, \min\{2, 1 + \theta(x)\})$, where*

$$\theta(x) = \left(\frac{1}{62} \log\left(1 + \frac{1 - ||x| - |x - e_1||}{4}\right)\right)^2.$$

Then

$$q(A) \leq 60\sqrt{e^{62\sqrt{K-1}} - 1}.$$

Proof. Let us first consider $|x| \le 3$. Now by Theorem 4.7 and Proposition 2.5

$$q(A) \le 2\mathrm{diam}\,(A) \le 8\sqrt{\varepsilon}(\min\{|x|,|x-e_1|\}+1) \le 30\sqrt{\varepsilon} \le 60\sqrt{e^{62\sqrt{K-1}}-1}.$$

Let us then consider $|x| > 3$. By Theorem 4.7 and Proposition 2.5 we have

$$\begin{aligned} q(A) &\le \frac{2\mathrm{diam}\,(A)}{1+(|x|-\varepsilon)^2} \le \frac{8\sqrt{\varepsilon}(\min\{|x|,|x-e_1|\}+1)}{1+(|x|-\varepsilon)^2} \\ &\le \frac{8\sqrt{\varepsilon}(3+|x|)}{1+|x|^2/4} \le 16\sqrt{\varepsilon} \le 32\sqrt{e^{62\sqrt{K-1}}-1}, \end{aligned}$$

where the fourth inequality follows from the fact that $(3+t)/(1+t^2/4) < 2$ for $t > 3$. The assertion follows from the above estimates. $\square$

5.4 Theorem. *(1) Let $r > 0$, $x \in B^3(r) \setminus \{0, e_1\}$, $\varepsilon \in (0, (1-||x|-|x-e_1||)/2)$, K be as in Lemma 4.3 and $f \in QC_K(\mathbb{R}^3)$ with $f(z) = z$ for $z \in \{0, e_1\}$. Then*

$$|p(f(x)) - p(x)| \le 4(r+1)\sqrt{\varepsilon}.$$

(2) Let $x \in \mathbb{R}^3 \setminus \{0, e_1\}$, $\varepsilon \in (0, (1-||x|-|x-e_1||)/2)$, K be as in Lemma 4.3 and $f \in QC_K(\mathbb{R}^3)$ with $f(z) = z$ for $z \in \{0, e_1\}$. Then

$$q(p(f(x)), p(x)) \le 12\sqrt{2}\sqrt{e^{62\sqrt{K-1}}-1},$$

where $p(x) = p((x_1, x_2, x_3)) = (x_1, \sqrt{x_2^2 + x_3^2}, 0)$.

Proof. (1) By Lemma 4.3 and Theorem 4.7 $|p(f(x))-p(x)| \le \sqrt{\varepsilon}4(\min\{|x|,|x-e_1|\}+1)$. Therefore

$$|p(f(x)) - p(x)| \le 4(r+1)\sqrt{\varepsilon}.$$

(2) We assume first that $|x| \le 2$. By definition of p we have $|p(x)| = |x|$ and by Lemma 4.3 we may assume $|p(f(x))| \ge |x|-\varepsilon$ and $\varepsilon = \exp(60\sqrt{K-1})-1$. Therefore by (1)

$$\begin{aligned} q(p(f(x)), p(x)) &= \frac{|p(f(x)) - p(x)|}{\sqrt{1+|p(f(x))|^2}\sqrt{1+|p(x)|^2}} \\ &\le \frac{12\sqrt{\varepsilon}}{1+(|x|-\varepsilon)^2} \\ &\le \frac{12\sqrt{\varepsilon}}{1+4\varepsilon} \\ &\le \frac{12\sqrt{2}\sqrt{e^{62\sqrt{K-1}}-1}}{8e^{62\sqrt{K-1}}-7} \\ &\le 12\sqrt{2}\sqrt{e^{62\sqrt{K-1}}-1}. \end{aligned}$$

Before considering the case $|x| > 2$, we note that for $s(z) = z/|z|^2$

$$s(p(x)) = \left(\frac{x_1}{|x|^2}, \frac{\sqrt{x_2^2 + x_3^2}}{|x|^2}, 0 \right) = p(s(x))$$

and for $g = s \circ f \circ s$ we have $g \in QC_K(\mathbb{R}^3)$ and $g(x) = x$ for $x \in \{0, e_1\}$.

If $|x| > 2$, then we have

$$\begin{aligned}
q(p(f(x)), p(x)) &= q(s(p(f(x))), s(p(x))) \\
&= q(p(s(f(x))), p(s(x))) \\
&= q(p(s(f(s(s(x))))), p(s(x))) \\
&= q(p(g(s(x))), p(s(x))) \\
&\leq 12\sqrt{2}\sqrt{e^{62\sqrt{K-1}} - 1},
\end{aligned}$$

because $|s(x)| < 1/2$. □

The following theorem is a special case of [S, Theorem 1.2]. However, the result is easy to generalize for arbitrary subdomains of $\mathbb{R}^n$.

5.5 Theorem. *Let $G = \mathbb{R}^n \setminus \{0\}$, $f \in QC_K(\mathbb{R}^n)$, $K \in (1, 2]$ and $f(0) = 0$. There exists $c(K)$ such that for all $x, y \in G$*

$$j_G(f(x), f(y)) \leq c(K) \max\{j_G(x, y)^\alpha, j_G(x, y)\},$$

where $\alpha = K^{1/(1-n)}$, and $c(K) \to 1$ as $K \to 1$.

Proof. Both sides of the claim are invariant under a homothety mapping $z \mapsto tz, t > 0$, and so is the normalization. Moreover, composition with a homothety does not change K-quasiconformality. Therefore we may assume that $x = e_1, f(e_1) = e_1$ and by symmetry we, furthermore, assume that $|y| \geq 1$. Now

$$\frac{|f(y) - f(e_1)|}{|f(e_1)|} = |f(y) - f(e_1)| \leq \eta\left(|e_1 - y|\right)$$

and

$$\frac{|f(y) - f(e_1)|}{|f(y)|} = \frac{|f(y) - f(e_1)|}{|f(y) - f(0)|} \leq \eta\left(\frac{|e_1 - y|}{|y - 0|}\right) = \eta\left(\frac{|e_1 - y|}{|y|}\right).$$

Therefore by Proposition 2.5 and Lemma 3.1 (7)

$$\begin{aligned} j(f(e_1), f(y)) &= \log\left(1 + \frac{|f(e_1) - f(y)|}{\min\{|f(e_1)|, |f(y)|\}}\right) \\ &\leq \log\left(1 + \max\left\{\eta(|y - e_1|), \eta\left(\frac{|x-y|}{|y|}\right)\right\}\right) \\ &= \log(1 + \eta(|y - e_1|)) \\ &\leq \log(1 + c_3 \max\{|y - e_1|^\alpha, |y - e_1|^{1/\alpha}\}) \\ &\leq \begin{cases} c_3 \log^\alpha(1 + |y - e_1|), & 0 < |y - e_1| < 1, \\ \frac{c_3}{\alpha} \log(1 + |y - e_1|), & |y - e_1| \geq 1. \end{cases} \end{aligned}$$

By choosing $c(K) = c_3/\alpha$, where $c_3 = \exp(60\sqrt{K-1})$ is as in Lemma 2.10, we have $c(K) \to 1$ as $K \to 1$ and the assertion follows. $\square$

5.6 Corollary. *Let $G \subset \mathbb{R}^n$ be a domain, $f \in QC_K(\mathbb{R}^n)$, $K \in (1, 2]$ and $f(z) = z$ for all $z \in \partial G$. There exists $c(K)$ such that for all $x, y \in G$*

$$j_G(f(x), f(y)) \leq c(K) \max\{j_G(x, y)^\alpha, j_G(x, y)\},$$

where $\alpha = K^{1/(1-n)}$, and $c(K) \to 1$ as $K \to 1$.

Proof. We may assume $0, e_1 \in \partial G$ and $d(x) = |x| \leq d(y) \leq |y|$. Now for $z \in \partial G$

$$\frac{|f(y) - f(x)|}{|z - f(x)|} = \frac{|f(y) - f(x)|}{|f(z) - f(x)|} \leq \eta\left(\frac{|x - y|}{|z - x|}\right) \leq \eta\left(\frac{|x - y|}{|x|}\right)$$

and

$$\frac{|f(y) - f(x)|}{|z - f(y)|} = \frac{|f(y) - f(x)|}{|f(z) - f(y)|} \leq \eta\left(\frac{|x - y|}{|z - y|}\right) \leq \eta\left(\frac{|x - y|}{|y|}\right) \leq \eta\left(\frac{|x - y|}{|x|}\right).$$

Therefore by Proposition 2.5 and Lemma 3.1 (7)

$$\begin{aligned} j(f(x), f(y)) &= \log\left(1 + \frac{|f(x) - f(y)|}{\min\limits_{z \in \partial G}\{|f(x) - z|, |f(y) - z|\}}\right) \\ &\leq \log\left(1 + \eta\left(\frac{|y - x|}{|x|}\right)\right) \\ &\leq \log\left(1 + c_3 \max\left\{\frac{|y - x|^\alpha}{|x|^\alpha}, \frac{|y - x|^{1/\alpha}}{|x|^{1/\alpha}}\right\}\right) \\ &\leq \begin{cases} c_3 \log^\alpha\left(1 + \frac{|y-x|}{|x|}\right), & 0 < |y - x| < 1, \\ \frac{c_3}{\alpha} \log\left(1 + \frac{|y-x|}{|x|}\right), & |y - x| \geq 1. \end{cases} \end{aligned}$$

By choosing $c(K) = c_3/\alpha$, where $c_3 = \exp(60\sqrt{K-1})$ is as in Lemma 2.10, we have $c(K) \to 1$ as $K \to 1$ and the assertion follows. $\square$

5.7 Theorem. *For given $K \in (1,2]$ and $n \geq 2$ there exists a constant $\omega(K,n)$ such that if $G = \mathbb{R}^n \setminus \{0\}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiconformal mapping with $f(0) = 0$, then for all $x, y \in G$*

$$k_G(f(x), f(y)) \leq \omega(K,n) \max\{k_G(x,y)^\alpha, k_G(x,y)\}$$

where $\alpha = K^{1/(1-n)}$ and $\omega(K,n) \to 1$ when $K \to 1$.

Proof. Fix $\lambda \in (0,1)$. Let $c \geq 1$ be the constant of Theorem 5.5 and write

$$\mu(\lambda) = \frac{1}{c} \min\{\log(1+\lambda), \log^\beta(1+\lambda)\} = \frac{1}{c} \log^\beta(1+\lambda)\,, \quad \beta = 1/\alpha\,.$$

Note that $c \to 1$ when $K \to 1$. The rest of the proof is now divided into two cases.

Case A. $k_G(x,y) \leq \mu(\lambda)$. By Theorem 5.5 and Lemma 5.1 we have

$$\begin{aligned}
\frac{|f(x)-f(y)|}{|f(x)|} &\leq \exp\left(c \max\{j_G(x,y)^\alpha, j_G(x,y)\}\right) - 1 \\
&\leq \exp\left(c \max\left\{\frac{\log^\beta(1+\lambda)}{c}, \frac{\log(1+\lambda)}{c}\right\}\right) - 1 \\
&= \lambda\,. \qquad (5.8)
\end{aligned}$$

Therefore we can use Lemma 5.1 and Theorem 5.5 to find an upper bound for $k_G(f(x), f(y))$ in terms of $k_G(x,y)$ and λ. By Lemma 5.1, (5.8), and Theorem 5.5 we have

$$\begin{aligned}
k_G(f(x), f(y)) &\leq c_1(\lambda) j_G(f(x), f(y)) \leq c c_1(\lambda) \max\{j_G(x,y)^\alpha, j_G(x,y)\} \\
&\leq c c_1(\lambda) \max\{k_G(x,y)^\alpha, k_G(x,y)\}\,.
\end{aligned}$$

Case B. $k_G(x,y) \geq \mu(\lambda)$. Choose points $x_j, j = 0, ..., p+1$, on the quasihyperbolic geodesic segment $J_G[x,y]$ joining x with y such that $x_0 = y, x_{p+1} = x$ and $k_G(x_{j+1}, x_j) = \mu(\lambda)$ for $j = 0, ..., p-1$, $k_G(x_{p+1}, x_p) \leq \mu(\lambda)$. Because the quasihyperbolic distance is additive on the geodesic we see that

$$k_G(x,y) = \sum_{j=0}^{p} k_G(x_{j+1}, x_j) \geq p\mu(\lambda) \Rightarrow p \leq k_G(x,y)/\mu(\lambda)\,.$$

Next, by the triangle inequality and by Case A

$$k_G(f(x), f(y)) \leq \sum_{j=0}^{p} k_G(f(x_{j+1}), f(x_j)) \leq c c_1(\lambda) \sum_{j=0}^{p} k_G(x_{j+1}, x_j)^\alpha = V\,.$$

Next, using the property of weighted mean values [M, p. 76, Theorem 1]

$$\sum_{j=0}^{p} u_j^{\alpha} \le (p+1)^{1-\alpha} \left(\sum_{j=0}^{p} u_j \right)^{\alpha} \quad \text{for all positive } u_j$$

we have

$$\begin{aligned} V &\le cc_1(\lambda)(p+1)^{1-\alpha} \left(\sum_{j=0}^{p} k_G(x_{j+1}, x_j) \right)^{\alpha} \\ &\le cc_1(\lambda)\mu(\lambda)^{\alpha-1} \left(1 + \frac{\mu(\lambda)}{k_G(x,y)} \right)^{1-\alpha} k_G(x,y) \\ &\le cc_1(\lambda)\mu(\lambda)^{\alpha-1} 2^{1-\alpha} k_G(x,y)\,. \end{aligned}$$

In view of the above Cases A and B we see that in both cases we have the claim with the constant

$$\omega(K,n) = cc_1(\lambda)\mu(\lambda)^{\alpha-1}2^{1-\alpha}\,.$$

It remains to prove that $\omega(K,n) \to 1$ when $K \to 1$. It suffices to show that we can choose λ depending on K, n such that

$$\mu(\lambda)^{\alpha-1} = c^{1-\alpha} \log^{1-\beta}(1+\lambda) \to 1$$

when $K \to 1$. For instance the choice $\lambda = \beta - 1$ will do. $\square$

Agard and Gehring have studied the angle distortion under quasiconformal mappings of the plane [AG]. Motivated by their work we record the following corollary of Theorem 5.7.

5.9 Corollary. *Suppose that under the hypotheses of Theorem 5.7 $x, y \in S^{n-1}$ and $f(x), f(y) \in S^{n-1}$ and let ϕ and ψ be the angles between the segments $[0,x], [0,y]$ and $[0,f(x)], [0,f(y)]$, respectively. Then*

$$\psi \le \omega(K,n) \max\{\phi^{\alpha}, \phi\}\,.$$

Proof. The proof follows easily from the Martin-Osgood formula (5.2). $\square$

5.10 Remark. *It is well-known (see [LVV],[AVV2, Lemma 4.28]) that for a given $K \ge 1$ there exists a K-quasiconformal mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$ such that $f(-e_1) = -e_1$ and $f(e_1) = \lambda(K^{1/(n-1)})e_1$, where $\lambda(K) = \varphi_K(1/\sqrt{2})^2/(1-\varphi_K(1/\sqrt{2})^2)$ (see [AVV, p. 203, 204]). Choosing $G = \mathbb{R}^n \setminus \{0\}$ and $x = 1 = -y$ in Theorem 5.5 we see that*

$$\log(1+L) \le c(K)\log 3,$$

where $L = (1 + \lambda(K^{1/(n-1)}))$*, and hence the constant* $c(K)$ *has to satisfy*

$$c(K) \geq \frac{\log(2 + \lambda(K^{1/(n-1)}))}{\log 3}.$$

In order to compare this estimate to the upper bound in Theorem 5.5 well-known estimates for $\lambda(K^{1/(n-1)})$ *may be used. For instance we know that* $\lambda(K) \geq \exp(\pi(K-1))$ *by [AVV, Corollary 10.33].*

This same idea can be applied to produce a lower bound for the constant $\omega(K, n)$ *of Theorem 5.7 as well.*

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